Khatra Adibasi Mahavidyalaya: Lecture Notes

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Hyperbolic Functions & applications

Definitions & identities

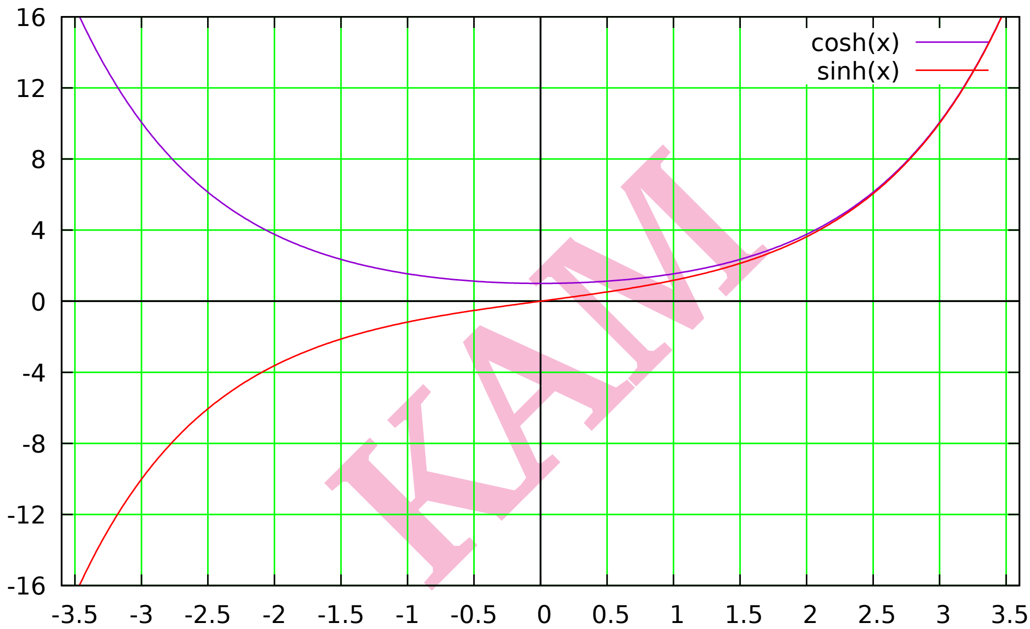

Figure 1: cosh & sinh plots. Note how their magnitudes approach each other for large x. Why? What will be the nature of $\tanh(x)$ plot?

The defining equations are:

$$\cosh(x) \equiv \frac{\exp(x) + \exp(-x)}{2}, \tag{1}$$

$$\sinh(x) \equiv \frac{\exp(x) - \exp(-x)}{2}, \tag{2}$$

$$\tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)}. \tag{3}$$

over the interval $-\infty < x < \infty$.
Using these equations, **prove** the following identities:

$$
\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= 1, \\
\cosh(2x) &= 2\cosh^2(x) - 1, \\
\cosh(2x) &= 2\sinh^2(x) + 1, \\
\sinh(2x) &= 2\sinh(x)\cosh(x), \\
\frac{d\cosh(x)}{dx} &= \sinh(x), \\
\frac{d\sinh(x)}{dx} &= \cosh(x), \\
\int \cosh(x)\,dx &= \sinh(x) + C, \\
\int \sinh(x)\,dx &= \cosh(x) + C.
\end{aligned}
$$

The *inverse* functions are defined as:

$$
\begin{aligned}
x &= \cosh(y) & \qquad x &= \sinh(y) \\
y &= \cosh^{-1}(x),\ x \geq 1 & \qquad y &= \sinh^{-1}(x),\ -\infty < x < \infty
\end{aligned}
$$

Note the *different* domains of the two inverse functions. See the graphs (Fig.1) of the hyperbolic functions for reasons.

Consider the first pair of equations above.

$$
\begin{aligned}
\exp(y) &= \cosh(y) + \sinh(y) \\
&= x + \sqrt{\cosh^2(y) - 1} \\
&= x + \sqrt{x^2 - 1} \\
\ln[\exp(y)] &= \ln\left(x + \sqrt{x^2 - 1}\right) \\
y &= \ln\left(x + \sqrt{x^2 - 1}\right) \\
\cosh^{-1}(x) &= \ln\left(x + \sqrt{x^2 - 1}\right),\ x \geq 1 \qquad (4)
\end{aligned}
$$

For the second pair, another useful identity results:

$$
\begin{aligned}
\exp(y) &= \cosh(y) + \sinh(y) \\
&= \sqrt{\sinh^2(y) + 1} + x \\
&= x + \sqrt{x^2 + 1} \\
\ln[\exp(y)] &= \ln\left(x + \sqrt{x^2 + 1}\right) \\
y &= \ln\left(x + \sqrt{x^2 + 1}\right) \\
\sinh^{-1}(x) &= \ln\left(x + \sqrt{x^2 + 1}\right),\ -\infty < x < \infty \qquad (5)
\end{aligned}
$$

Note that the argument for the logarithm is *always* ≥ 0, hence no modulus sign is required. **Prove** that:

$$\cosh^{-1}(x) = \sinh^{-1}(\sqrt{x^2-1}),\ x \geq 1 \tag{6}$$

$$\sinh^{-1}(x) = \cosh^{-1}(\sqrt{x^2+1}),\ -\infty < x < \infty \tag{7}$$

Compare eqns 6 & 7 with the corresponding relations for the inverse trigonometric functions $\sin^{-1} x$ & $\cos^{-1} x$.

Evaluation of integrals

Evaluation of $\int \frac{dx}{\sqrt{x^2+a^2}}$: For $y = x/a$, we have

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{dy}{\sqrt{y^2+1}} \\
&= \int \frac{\cosh(z)\,dz}{\cosh(z)},\ \text{for}\ y = \sinh(z) \\
&= z + C\ = \sinh^{-1}(y) + C
\end{aligned}$$

$$\text{or,} \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a) + C\ = \ln\left(x + \sqrt{x^2+a^2}\right) + C', \tag{8}$$

where eq.5 has been used ($C' = C - \ln a$).

Following the preceding steps, **prove** that:

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + C\ = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \tag{9}$$

Prove that:

$$\begin{aligned}
\int \sqrt{1+a^2\,x^2}\,dx &= \frac{1}{2a}\sinh^{-1}(ax) + \frac{x}{2}\sqrt{1+a^2\,x^2} + C \\
&= \frac{1}{2a}\ln\left(ax + \sqrt{1+a^2\,x^2}\right) + \frac{x}{2}\sqrt{1+a^2\,x^2} + C
\end{aligned} \tag{10}$$

Hints: Make the substitutions $y = \sqrt{1+a^2\,x^2}$ & then $z = \cosh^{-1}(y)$. This reduces the integral to $(1/2a)\int(\cosh(2z)+1)dz$. After integration, use the various identites to simplify & obtain the final expressions.

The Hyperbola

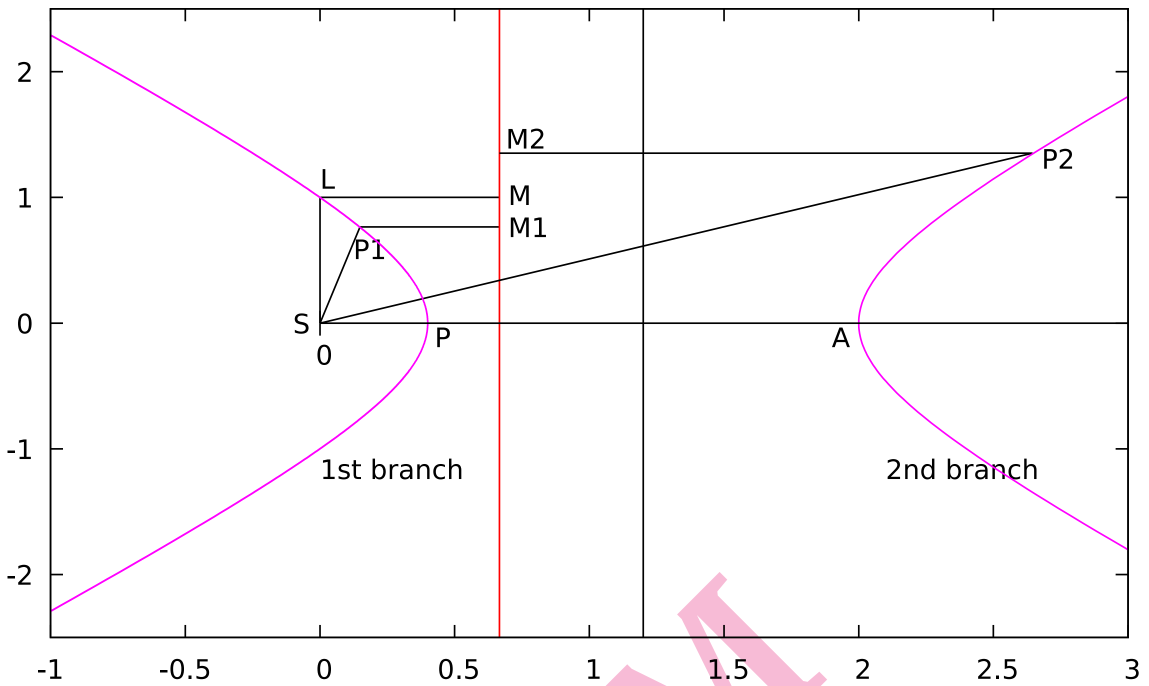

Figure 2: Polar plots for the two branches of a hyperbola. $l = 1$; $e = 1.5$. The origin is the first focus S & the red line is the corresponding directrix.

The equation of any conic section in plane-polar coordinates is given by

$$r = \frac{l}{1 + e\,\cos\phi}, \tag{11}$$

where $\phi = 0$ is the position of the periapsis (defined later).

In plane geometry, a conic section is defined as the locus of points whose distances to a fixed point (the *focus*) and a fixed line (the *directrix*) always has the same ratio (the *eccentricity* e). If $P1$ be a point on the conic, then $SP1 = e\,P1M1$ (see Fig.2). The coordinates of $P1$ is (r, ϕ), $\phi = 0$ being along SA. The length of a chord parallel to the directrix & passing through the focus is called the *latus rectum* $2\,l$ ($SL = l$ in Fig.2). Then $SL = e\,LM$, by definition. From the figure, $LM = SP1\cos\phi + P1M1 = r\cos\phi + P1M1$.

Hence,

$$LM = \frac{l}{e} = r\cos\phi + \frac{r}{e}, \ \text{ which is Eq.11.}$$

The periapsis & apoapsis are the nearest & farthest points on the conic from the focus. For $e < 1$, an ellipse, the periapsis lies between the focus & its directrix. The apoapsis lies in the side of the focus opposite to the directrix.

The parabola & hyperbola have no apoapsis. However, the hyperbola, for which $e > 1$, there is a **second branch** on the opposite side of the directrix. This branch also has a periapsis, marked A in Fig.2. For any point $P2$ on this branch, we must have

$SP2 = e\,P2M2$. Let the coordinates of $P2$ be (r', ϕ'). From the figure, $LM + P2M2 = SP2 \cos\phi' = r' \cos\phi'$. Hence,

$$LM + P2M2 = \frac{l}{e} + \frac{r'}{e} = r' \cos\phi'$$

On rearrangement, the equation for the second branch is

$$r' = \frac{l}{e\,\cos\phi' - 1}, \tag{12}$$

where (r', ϕ') are the polar coordinates of points on the second branch with the **first** focus as the origin. The two periapses distances are therefore,

$$r_1 = \frac{l}{e+1} \qquad\qquad r_2 = \frac{l}{e-1} \tag{13}$$

The major axis for a hyperbola $= 2a$ is defined as the distance between the two periapses (AP in Fig.2). Or,

$$a = \frac{r_2 - r_1}{2} = \frac{l}{e^2 - 1} \tag{14}$$

A second symmetry axis of the hyperbola, called the minor axis, also exists which is perpendicular to the major axis & lies midway between A and P in Fig.2, or midway between the **second** & first foci (see Fig.3). Now, consider any point P on a single branch of a hyperbola. W.r.t. its first focus S, the equation of the branch is given by Eq.11, with (r, ϕ) being the coordinates of P with S as origin & $\phi = 0$ is along SD, ϕ increases in the *anti-clockwise* direction. If the second focus S' is taken as the origin, the coordinates of P are $(r'\phi')$, with $\phi' = 0$ along $S'D'$, ϕ' increases in the *clockwise* direction . The equation of this branch is therefore given by Eq.12.

From Fig.3, $SS' = r_1 + r_2 = 2le/(e^2 - 1)$.

Also, $SS' = SQ + QS' = SP\cos\phi + S'P\cos\phi' = r\cos\phi + r'\cos\phi'$. Using Eqns. 11 & 12, we have

$$\frac{2le}{e^2-1} = \frac{l-r}{e} + \frac{l+r'}{e} = \frac{2l}{e} + \frac{r'-r}{e}$$

$$\text{or, } r' - r = \frac{2le^2}{e^2-1} - 2l = \frac{2l}{e^2-1} = 2a, \ \ (\text{see Eq.14}) \tag{15}$$

Eq.15 is satisfied by any point on a particular branch of a hyperbola, r being its distance from the first focus & r', the distance from the second focus.

Hyperbola in Cartesian coordinates

A hyperbola, with its focus on the $X-$axis is described by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{16}$$

where a & b are its semi-major & semi-minor axis distances respectively. The slope of the curve is $dy/dx = (x/y)(b^2/a^2)$. In the asymptotic region, Eq.16 reduces to

$$\frac{x^2}{a^2} \approx \frac{y^2}{b^2}$$

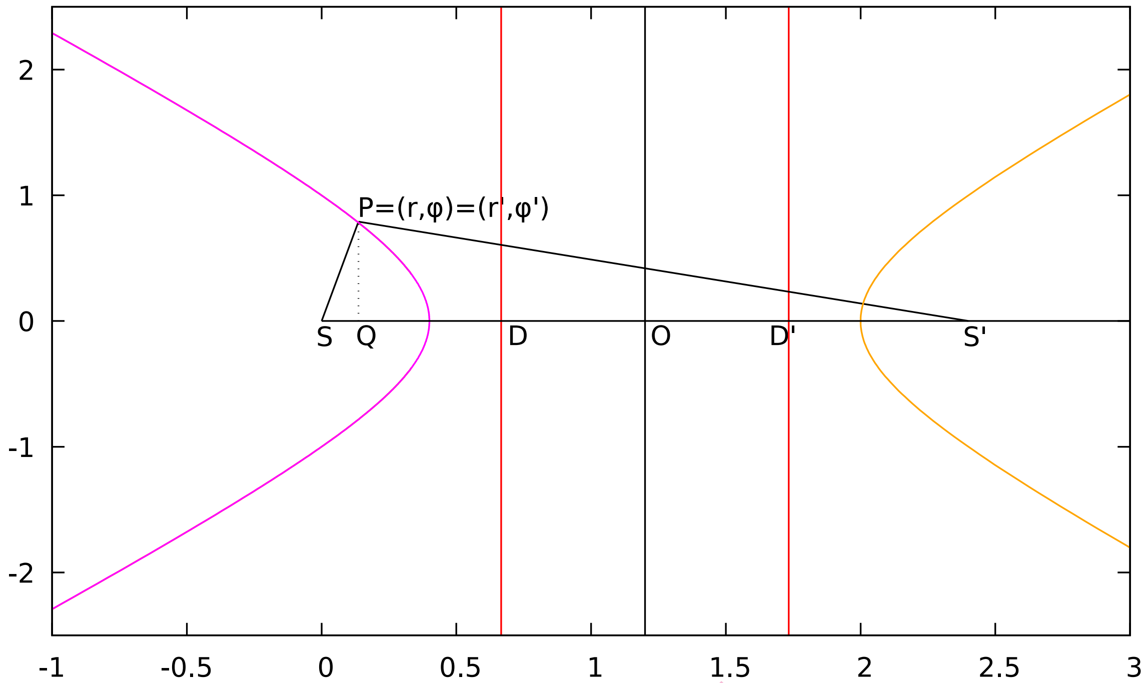

Figure 3: A point P on a single branch has coordinates (r, ϕ) & $(r'\phi')$ w.r.t. the two foci S & S' as origins respectively. $l = 1$; $e = 1.5$. For each focus, there is a corresponding directrix.

Hence, the slope of the asymptote (which is equal to the slope of the curve in the asymptotic region) is given by

$$\frac{dy}{dx} = \pm\frac{a}{b}\frac{b^2}{a^2} = \pm\frac{b}{a} \tag{17}$$

Therefore, the height of a triange with a as its base & the asymptote of slope b/a as hypotenuse, is simply b (see Fig.4)

Relationship of a, b & e:

The eccentricity of the hyperbola can be obtained by comparing the Cartesian & the polar form. The branch located at $x \geq a$ is given in the polar form by Eq.12. The polar angle of the asymptote is obtained by taking the limit $r \to \infty$ or $\phi' \to \cos^{-1}(1/e)$.

Hence $\tan^{-1}(b/a) = \cos^{-1}(a/\sqrt{a^2 + b^2}) = \cos^{-1}(1/e)$. Therefore we have

$$e = \sqrt{1 + \frac{b^2}{a^2}} \qquad\qquad b = a\sqrt{e^2 - 1} \tag{18}$$

The foci are located at a distance $\pm(r_1 + a) = \pm ae$ from the origin (see Fig.4 & Eq.14).

Rectangular Hyperbola

A hyperbola, with its major & minor axes of equal length, is called a rectangular hyperbola. The asymptotes, therefore, make an angle of $\tan^{-1}(1) = \pi/4$ with the symmetry axis & are hence mutually perpendicular. When the asymptotes are used as coordinate axes, the equation of the hyperbola takes a simple form.

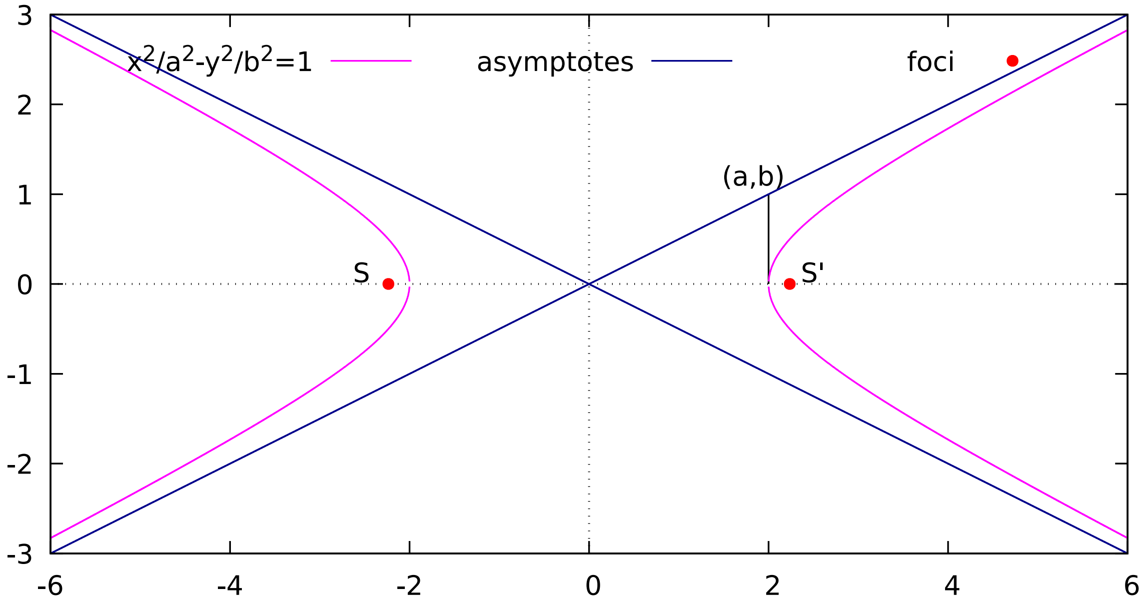

Figure 4: Hyperbola in Cartesian coordinates. $a = 2$; $b = 1$. For this choice, $e \approx 1.118$

Consider the rectangular hyperbola $y^2 - x^2 = a^2$, with its foci lying along the Y axis. To use the asymptotes as axes, an anti-clockwise rotation of angle $\pi/4$ about the Z axis is necessary. Let us call this system the $X'Y$" system. The coordinates of any point (x, y) transforms to (x', y'), by the transformation relations:

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta,$$

with $\theta = \pi/4$. Substituting in the relation $y^2 - x^2 = a^2$, we get the equation of a reactangular hyperbola, with asymptotes as axes, as

$$x'y' = \frac{a^2}{2}. \tag{19}$$

Note that the eccentricity of such a hyperbola is, by Eq.18, $\sqrt{2}$.