Khatra Adibasi Mahavidyalaya: Lecture Notes

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Vector Identities

Product Rules

For the del vector operator ($\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$) acting on the various combinations of the product of two *types* of fields $\boldsymbol{A(r)}$ & $f(\boldsymbol{r})$ (vector & scalar) the following vector identities are given, along with their proofs. The quantities are evaluated in Cartesian coordinates, the coordinate axes are chosen such that the vectors $\boldsymbol{A(r)}$ & $\boldsymbol{B(r)}$ lie in a plane perpendicular to $\hat{\boldsymbol{z}}$. Further, a rotation about $\hat{\boldsymbol{z}}$ is performed to make $\boldsymbol{A(r)} \,||\, \hat{\boldsymbol{x}}$.

Thus $\boldsymbol{A(r)} = A_x(x,y,z)\hat{\boldsymbol{x}}$ & $\boldsymbol{B(r)} = B_x(x,y,z)\hat{\boldsymbol{x}} + B_y(x,y,z)\hat{\boldsymbol{y}}$. This reduces the following quantities to a simplified form:

$$\boldsymbol{\nabla}.\boldsymbol{A} = \frac{\partial}{\partial x}A_x \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{\partial}{\partial z}A_x\hat{\boldsymbol{y}} - \frac{\partial}{\partial y}A_x\hat{\boldsymbol{z}} \tag{2}$$

$$\boldsymbol{\nabla}.\boldsymbol{B} = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y \tag{3}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = -\frac{\partial}{\partial z}B_y\hat{\boldsymbol{x}} + \frac{\partial}{\partial z}B_x\hat{\boldsymbol{y}} + \left(\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x\right)\hat{\boldsymbol{z}} \tag{4}$$

$$\boldsymbol{A}.\boldsymbol{B} = A_xB_x \tag{5}$$

$$\boldsymbol{A}\times\boldsymbol{B} = A_xB_y\hat{\boldsymbol{z}} \tag{6}$$

$$\boldsymbol{A}.\boldsymbol{\nabla} = A_x\frac{\partial}{\partial x} \tag{7}$$

$$\boldsymbol{B}.\boldsymbol{\nabla} = B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} \tag{8}$$

Note that $\boldsymbol{A}.\boldsymbol{\nabla}$ & $\boldsymbol{B}.\boldsymbol{\nabla}$ are *scalar* operators.

There are two ways to form a scalar using product of two types of fields: $f(\boldsymbol{r})\,g(\boldsymbol{r})$ & $\boldsymbol{A}.\boldsymbol{B}$. The corresponding two *gradient* rules are:

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f \tag{9}$$

$$\boldsymbol{\nabla}(\boldsymbol{A}.\boldsymbol{B}) = \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + (\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} \tag{10}$$

A vector is also formed in two ways: $f\boldsymbol{A}$ & $\boldsymbol{A}\times\boldsymbol{B}$. The corresponding two *divergence* rules are:

$$\boldsymbol{\nabla}.(f\boldsymbol{A}) = f\boldsymbol{\nabla}.\boldsymbol{A} + \boldsymbol{\nabla}f.\boldsymbol{A} \tag{11}$$

$$\boldsymbol{\nabla}.(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}.(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}.(\boldsymbol{\nabla}\times\boldsymbol{B}) \tag{12}$$

Note that in the last equation all quantities are scalar triple products & hence, the brackets are not *essential*.

The corresponding two *curl* rules are:

$$\boldsymbol{\nabla}\times(f\boldsymbol{A}) = f\,(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{\nabla} f\times\boldsymbol{A} \tag{13}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{\nabla}.\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{\nabla}.\boldsymbol{A})\boldsymbol{B} \tag{14}$$

In the last equation, only the bracket in the LHS is *essential*.

Proofs

1. *Proof* of Eq.(9): do it yourself.

2. *Proof* of Eq.(11): $\boldsymbol{\nabla}.(f\boldsymbol{A}) = \boldsymbol{\nabla}.(fA_x\hat{\boldsymbol{x}}) = \frac{\partial}{\partial x}(fA_x) = f\frac{\partial}{\partial x}A_x + A_x\frac{\partial}{\partial x}f = RHS$, using Eq.(1) & Eq.(5) in the last step, with $\boldsymbol{B}$ replaced by $\boldsymbol{\nabla} f$.

3. *Proof* of Eq.(13): Using Eq.(2) with $\boldsymbol{A}$ replaced by $f\boldsymbol{A}$,

$$LHS = \frac{\partial}{\partial z}(fA_x)\hat{\boldsymbol{y}} - \frac{\partial}{\partial y}(fA_x)\hat{\boldsymbol{z}} = f\left(\frac{\partial}{\partial z}A_x\hat{\boldsymbol{y}} - \frac{\partial}{\partial y}A_x\hat{\boldsymbol{z}}\right) + A_x\left(\frac{\partial}{\partial z}f\hat{\boldsymbol{y}} - \frac{\partial}{\partial y}f\hat{\boldsymbol{z}}\right) \tag{15}$$

   The sum of the first two terms of RHS of Eq.(15) is $f(\boldsymbol{\nabla}\times\boldsymbol{A})$, using Eq.(2). Evaluate $\boldsymbol{\nabla} f\times\boldsymbol{A}$ separately to show that it is the sum of the last two terms of Eq.(15). This proves Eq.(13).

4. *Proof* of Eq.(12):

$$\boldsymbol{\nabla}.(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{\nabla}.(A_xB_y\hat{\boldsymbol{z}}) = \frac{\partial}{\partial z}(A_xB_y) = B_y\frac{\partial}{\partial z}A_x + A_x\frac{\partial}{\partial z}B_y$$

$$\text{From Eq(2)}: \boldsymbol{B}.(\boldsymbol{\nabla}\times\boldsymbol{A}) = B_y\frac{\partial}{\partial z}A_x,$$

$$\text{from Eq(4)}: \boldsymbol{A}.(\boldsymbol{\nabla}\times\boldsymbol{B}) = -A_x\frac{\partial}{\partial z}B_y.$$

   This proves Eq(12).

5. *Proof* of Eq.(14):

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) &= (\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}) \times (A_x B_y \hat{\boldsymbol{z}}) = \frac{\partial}{\partial x}(A_x B_y)(-\hat{\boldsymbol{y}}) + \frac{\partial}{\partial y}(A_x B_y)(\hat{\boldsymbol{x}}) \\
&= -B_y \frac{\partial}{\partial x} A_x \hat{\boldsymbol{y}} - A_x \frac{\partial}{\partial x} B_y \hat{\boldsymbol{y}} + B_y \frac{\partial}{\partial y} A_x \hat{\boldsymbol{x}} + A_x \frac{\partial}{\partial y} B_y \hat{\boldsymbol{x}} \\
&= -B_y \hat{\boldsymbol{y}}(\boldsymbol{\nabla}.\boldsymbol{A}) - (\boldsymbol{A}.\boldsymbol{\nabla}) B_y \hat{\boldsymbol{y}} + B_y \frac{\partial}{\partial y}\boldsymbol{A} + \boldsymbol{A}(\frac{\partial}{\partial y} B_y)(\text{ see Eqns.1 \& 7}) \\
&= -(\boldsymbol{B} - B_x \hat{\boldsymbol{x}})(\boldsymbol{\nabla}.\boldsymbol{A}) - (\boldsymbol{A}.\boldsymbol{\nabla})(\boldsymbol{B} - B_x \hat{\boldsymbol{x}}) + (\boldsymbol{B}.\boldsymbol{\nabla} - B_x \frac{\partial}{\partial x})\boldsymbol{A} \\
&\quad + \boldsymbol{A}(\boldsymbol{\nabla}.\boldsymbol{B} - \frac{\partial}{\partial x} B_x)(\text{ see Eqns.3 \& 8}) \\
&= \{-\boldsymbol{B}(\boldsymbol{\nabla}.\boldsymbol{A}) - (\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A}(\boldsymbol{\nabla}.\boldsymbol{B})\} \\
&\quad + \left\{ B_x \hat{\boldsymbol{x}}(\boldsymbol{\nabla}.\boldsymbol{A}) + (\boldsymbol{A}.\boldsymbol{\nabla}) B_x \hat{\boldsymbol{x}} - B_x \frac{\partial}{\partial x}\boldsymbol{A} - \boldsymbol{A}\frac{\partial}{\partial x} B_x \right\} \\
&= \{(\boldsymbol{\nabla}.\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{\nabla}.\boldsymbol{A})\boldsymbol{B} + (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B}\} \\
&\quad + \left\{ B_x \hat{\boldsymbol{x}}(\boldsymbol{\nabla}.\boldsymbol{A}) + (\boldsymbol{A}.\boldsymbol{\nabla}) B_x \hat{\boldsymbol{x}} - B_x \hat{\boldsymbol{x}}\frac{\partial}{\partial x} A_x - A_x \frac{\partial}{\partial x}(B_x \hat{\boldsymbol{x}}) \right\}
\end{aligned}$$

In the last step, $\frac{\partial}{\partial x}(\hat{\boldsymbol{x}}) = 0$ has been used to shift $\hat{\boldsymbol{x}}$ to left or right of $\frac{\partial}{\partial x}$. RHS of Eq.(14) is already obtained as $-B_x \hat{\boldsymbol{x}} \frac{\partial}{\partial x} A_x = -B_x \hat{\boldsymbol{x}}(\boldsymbol{\nabla}.\boldsymbol{A})$ & $-A_x \frac{\partial}{\partial x}(B_x \hat{\boldsymbol{x}}) = -(\boldsymbol{A}.\boldsymbol{\nabla})(B_x \hat{\boldsymbol{x}})$, thus reducing the second group of terms within {} to zero.

6. *Proof* of Eq.(10): The LHS of Eq.(11) is expressed as:

$$\boldsymbol{\nabla}(\boldsymbol{A}.\boldsymbol{B}) = \boldsymbol{\nabla}(A_x B_x) = A_x \boldsymbol{\nabla} B_x + B_x \boldsymbol{\nabla} A_x, \tag{16}$$

using Eq.(5) & Eq.(9). Now, using Eqn(4), show that

$$\boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = -A_x \left(\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x \right) \hat{\boldsymbol{y}} + \left(A_x \frac{\partial}{\partial z} B_x \right) \hat{\boldsymbol{z}}$$

$$\begin{aligned}
\boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) &= \left(-A_x \frac{\partial}{\partial x} \right)(B_y \hat{\boldsymbol{y}}) + A_x \left(\hat{\boldsymbol{y}} \frac{\partial}{\partial y} B_x \right) + A_x \left(\hat{\boldsymbol{z}} \frac{\partial}{\partial z} B_x \right) \\
&= \left(-A_x \frac{\partial}{\partial x} \right)(\boldsymbol{B} - B_x \hat{\boldsymbol{x}}) + A_x \left(\hat{\boldsymbol{y}} \frac{\partial}{\partial y} B_x + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} B_x \right) \\
&= -(\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + A_x \left(\hat{\boldsymbol{x}} \frac{\partial}{\partial x} B_x \right) + A_x \left(\hat{\boldsymbol{y}} \frac{\partial}{\partial y} B_x + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} B_x \right) \\
&= -(\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + A_x \left(\hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) B_x \\
&= -(\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + A_x \boldsymbol{\nabla} B_x
\end{aligned}$$

Similarly, use Eqn(2) to show that

$$\boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \left(-B_y \frac{\partial}{\partial y} A_x \right) \hat{\boldsymbol{x}} + \left(B_x \frac{\partial}{\partial y} A_x \right) \hat{\boldsymbol{y}} + \left(B_x \frac{\partial}{\partial z} A_x \right) \hat{\boldsymbol{z}}$$

$$\begin{aligned}
\therefore\ \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) &= \left(-B_y\frac{\partial}{\partial y}\right)(A_x\hat{\boldsymbol{x}}) + B_x\left(\hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)A_x + B_x\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)A_x \\
&= \left(-\boldsymbol{B}.\boldsymbol{\nabla} + B_x\frac{\partial}{\partial x}\right)(A_x\hat{\boldsymbol{x}}) + B_x\left(\hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)A_x \\
&= -(\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} + B_x\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x}\right)A_x + B_x\left(\hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)A_x \\
&= -(\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} + B_x\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)A_x \\
&= -(\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} + B_x\boldsymbol{\nabla}A_x
\end{aligned}$$

[*N.B.* from the expressions for $\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})$, the expression for $\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A})$ *cannot* be obtained by simply interchanging $\boldsymbol{A}$ & $\boldsymbol{B}$. The fact that the expressions are obtained from each other by interchange of *symbols* A & B is a peculiarity of the particular orientation choice of our coordinate system.]

Now, add the two expressions to get

$$\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = -(\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + A_x\boldsymbol{\nabla}B_x - (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} + B_x\boldsymbol{\nabla}A_x$$

Rearranging terms & using Eq.(16), we have

$$\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + (\boldsymbol{A}.\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} = \boldsymbol{\nabla}(A_xB_x) = \boldsymbol{\nabla}(\boldsymbol{A}.\boldsymbol{B})$$

Second Derivatives

The operator $\boldsymbol{\nabla}$ acting on the scalar field $\boldsymbol{\nabla}.\boldsymbol{A}$ & the two vector fields $\boldsymbol{\nabla} f$ & $\boldsymbol{\nabla}\times\boldsymbol{A}$ gives rise to five identities:

1. $\boldsymbol{\nabla}.(\boldsymbol{\nabla} f) = \boldsymbol{\nabla}^2 f$, the Laplacian of f, which is a scalar field. This is the defining equation for the Laplacian.

2. $\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0$. This is easily proved, assuming equality of mixed second derivative operators like $\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$.

3. $\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A})$. This term appears in Eq.(17).

4. $\boldsymbol{\nabla}.(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0$. Again, the proof is similar to item 2.

5. $$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A} \tag{17}$$

   This is also the defining equation for the Laplacian of a vector field.

   Proof:

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}\times\left(\frac{\partial}{\partial z}A_x\hat{\boldsymbol{y}} - \frac{\partial}{\partial y}A_x\hat{\boldsymbol{z}}\right) &= \hat{\boldsymbol{x}}\left(-\frac{\partial^2}{\partial y^2}A_x - \frac{\partial^2}{\partial z^2}A_x\right) \\ &+ \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\frac{\partial}{\partial y}A_x + \hat{\boldsymbol{z}}\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_x (\text{from Eq.(2)}) \\ &= \hat{\boldsymbol{x}}\left(-\frac{\partial^2}{\partial x^2}A_x - \frac{\partial^2}{\partial y^2}A_x - \frac{\partial^2}{\partial z^2}A_x\right) + \\ &\hat{\boldsymbol{x}}\frac{\partial^2}{\partial x^2}A_x + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\frac{\partial}{\partial x}A_x + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\frac{\partial}{\partial x}A_x \\ &= -\boldsymbol{\nabla}^2(A_x\hat{\boldsymbol{x}}) + \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}\end{aligned}$$

   Note that $\boldsymbol{\nabla}\times\boldsymbol{\nabla}$ is a null operator.

Integral Theorems

We use $d\,\boldsymbol{r}$ to denote an infinitesial displacement in line integrals, $d\,\sigma$ & $d\,\tau$ as infinitesimal surface & volume elements respectively, all located at $\boldsymbol{r}$. $\hat{\boldsymbol{n}}$ will denote the unit normal at the location of $d\,\sigma$. The gradient, divergence & Stokes's theorem are respectively:

$$\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b}\boldsymbol{\nabla} f(\boldsymbol{r}).d\,\boldsymbol{r} = f(\boldsymbol{r}_b) - f(\boldsymbol{r}_a) \tag{18}$$

$$\int_\tau \boldsymbol{\nabla}.\boldsymbol{A}(\boldsymbol{r})\,d\,\tau = \oint_\sigma \boldsymbol{A}(\boldsymbol{r}).\hat{\boldsymbol{n}}\,d\,\sigma \tag{19}$$

$$\int_\sigma \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}).\hat{\boldsymbol{n}}\,d\,\sigma = \oint_\gamma \boldsymbol{A}(\boldsymbol{r}).d\,\boldsymbol{r} \tag{20}$$

The LHS of the gradient theorem is an integral along an *open* curve whose boundaries are the *two* points $\boldsymbol{r}_a$ & $\boldsymbol{r}_b$. For a fixed pair of end-points, the integral is independent of the choice of the curve joning them.

The LHS of the divergence theorem is an integral over a volume τ whose boundary is the *closed* surface σ. For a closed surface, $\hat{\boldsymbol{n}}$ is always directed away from the volume.

The LHS of Stoke's theorem is an integral over an *open* surface σ whose boundary is the *closed* curve γ. The direction in which the line integral is performed is given by the right-hand-rule, if the thumb points along $\hat{\boldsymbol{n}}$. Conversely, the orientation of σ, *i.e.*, the direction of $\hat{\boldsymbol{n}}$ is given by the right-hand-rule, if the four fingers curl in the direction of traversal of γ. For a fixed boundary γ, the surface integral is independent of the choice of the surface enclosed by γ.

All three theorems are higher dimensional forms of the fundamental theorem of calculus:

$$\int_a^b \frac{d\,F(x)}{d\,x}\,d\,x = F(b) - F(a), \tag{21}$$

i.e., the integral of a derivative of a function over a region is given by its values at the region's boundaries.

Corollaries

$$\int_\tau \boldsymbol{\nabla} f\,d\,\tau = \oint_\sigma f\,\hat{\boldsymbol{n}}\,d\,\sigma \tag{22}$$

$$\int_\tau (\boldsymbol{\nabla}\times\boldsymbol{A})\;d\,\tau = \oint_\sigma (\hat{\boldsymbol{n}}\times\boldsymbol{A})\;d\,\sigma \tag{23}$$

$$\int_\sigma (\hat{\boldsymbol{n}}\,\times\boldsymbol{\nabla} f).d\,\sigma = \oint_\gamma f\,d\,\boldsymbol{r} \tag{24}$$

$$\int_\sigma [(\hat{\boldsymbol{n}}\,\times\boldsymbol{\nabla})\times\boldsymbol{A}]\,d\,\sigma = \oint_\gamma d\,\boldsymbol{r}\times\boldsymbol{A} \tag{25}$$

1. *Proof of Eq(22):* Take $\boldsymbol{A} = f\,\boldsymbol{B}$, with $\boldsymbol{B}$ a constant vector, in the divergence theorem. The product rule in Eq.(11) gives $\boldsymbol{\nabla}.\boldsymbol{A} = \boldsymbol{\nabla} f.\boldsymbol{B}$. Hence

   $$\left(\int_\tau \boldsymbol{\nabla} f\,d\,\tau\right).\boldsymbol{B} = \left(\oint_\sigma f\,\hat{\boldsymbol{n}}\,d\,\sigma\right).\boldsymbol{B},$$

   for any *arbitrary* constant vector $\boldsymbol{B}$. Choosing $\boldsymbol{B} = \hat{\boldsymbol{x}}$ proves the equality of the x components of the two sides of Eq.(22). Proceeding similarly for other components, the relation is proved.

   For $f(\boldsymbol{r}) =$ *constant*, the LHS vanishes & we get that the *total vector area for a closed surface is zero*:

   $$\oint_\sigma \hat{\boldsymbol{n}}\,d\,\sigma = 0 \tag{26}$$

2. *Proof of Eq(23):* Take $\boldsymbol{C} = \boldsymbol{A}\times\boldsymbol{B}$, with $\boldsymbol{B}$ a constant vector, in the divergence theorem. The product rule in Eq.(12) gives $\boldsymbol{\nabla}.\boldsymbol{C} = \boldsymbol{B}.(\boldsymbol{\nabla}\times\boldsymbol{A})$. Hence

   $$\left(\int_\tau \boldsymbol{\nabla}\times\boldsymbol{A}\,d\,\tau\right).\boldsymbol{B} = \oint_\sigma (\boldsymbol{A}\times\boldsymbol{B})\,.\hat{\boldsymbol{n}}\,d\,\sigma, = \left(\oint_\sigma \hat{\boldsymbol{n}}\times\boldsymbol{A}\;d\,\sigma\right).\boldsymbol{B},$$

where in the last step, the vectors are cyclically rearranged in the scalar triple product. Again, this is true for any *arbitrary* constant vector $\boldsymbol{B}$, which proves Eq.(23). Taking $\boldsymbol{A} = constant$ yields Eq.(26) again.

3. *Proof of Eq(24):* Take $\boldsymbol{A} = f\,\boldsymbol{B}$, with $\boldsymbol{B}$ a constant vector, in Stoke's theorem. The product rule in Eq.(13) gives $\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} f \times \boldsymbol{B}$. Hence

$$\int_\sigma (\boldsymbol{\nabla} f \times \boldsymbol{B}).\hat{\boldsymbol{n}}\, d\,\sigma = \oint_\gamma f\,\boldsymbol{B}.d\,\boldsymbol{r},$$
$$\text{or,} \left(\int_\sigma (\hat{\boldsymbol{n}} \times \boldsymbol{\nabla} f)\, d\,\sigma\right).\boldsymbol{B} = \left(\oint_\gamma f\, d\,\boldsymbol{r}\right).\boldsymbol{B},$$

where a cyclic rearrangement of the scalar triple product has been made in the RHS. This proves Eq.(24).

For $f(\boldsymbol{r}) = constant$, the LHS vanishes & we get that the *total vector displacement for a closed curve is zero*:

$$\oint_\gamma d\,\boldsymbol{r} = 0 \tag{27}$$

4. *Proof of Eq(25)*(optional): Take $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$, with $\boldsymbol{B}$ a constant vector, in Stoke's theorem. The product rule in Eq.(14) gives $\boldsymbol{\nabla} \times \boldsymbol{C} = (\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{\nabla}.\boldsymbol{A})\boldsymbol{B}$. Hence

$$\int_\sigma \hat{\boldsymbol{n}}.\left[\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B})\right] d\,\sigma = \oint_\gamma (\boldsymbol{A} \times \boldsymbol{B}).d\,\boldsymbol{r}$$
$$\text{or} \int_\sigma \hat{\boldsymbol{n}}.\left[(\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{\nabla}.\boldsymbol{A})\boldsymbol{B}\right] d\,\sigma = \oint_\gamma (d\,\boldsymbol{r} \times \boldsymbol{A}).\boldsymbol{B} \tag{28}$$

Now, if $\hat{\boldsymbol{n}} = (n_x, n_y, n_z)$, then the vector operator $\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}$ is given by

$$\hat{\boldsymbol{n}} \times \boldsymbol{\nabla} = \hat{\boldsymbol{x}}\left(n_y \frac{\partial}{\partial\, z} - n_z \frac{\partial}{\partial\, y}\right) + \hat{\boldsymbol{y}}\left(n_z \frac{\partial}{\partial\, x} - n_x \frac{\partial}{\partial\, z}\right) + \hat{\boldsymbol{z}}\left(n_x \frac{\partial}{\partial\, y} - n_y \frac{\partial}{\partial\, x}\right)$$

Now, for the coordinate system oriented such that $\boldsymbol{A} = A_x\hat{\boldsymbol{x}}$, we have

$$
\begin{aligned}
(\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}) \times \boldsymbol{A} &= (-\hat{\boldsymbol{z}}) \left(n_z \frac{\partial}{\partial x} A_x - n_x \frac{\partial}{\partial z} A_x \right) + (\hat{\boldsymbol{y}}) \left(n_x \frac{\partial}{\partial y} A_x - n_y \frac{\partial}{\partial x} A_x \right) \\
&= \left[n_x \left(\hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) A_x + n_x \left(\hat{\boldsymbol{y}} \frac{\partial}{\partial y} \right) A_x \right] - (n_y \hat{\boldsymbol{y}} + n_z \hat{\boldsymbol{z}}) \left(\frac{\partial}{\partial x} A_x \right) \\
&= \left[n_x \left(\hat{\boldsymbol{x}} \frac{\partial}{\partial x} \right) A_x + n_x \left(\hat{\boldsymbol{y}} \frac{\partial}{\partial y} \right) A_x + n_x \left(\hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) A_x \right] \\
&\quad - (n_x \hat{\boldsymbol{x}} + n_y \hat{\boldsymbol{y}} + n_z \hat{\boldsymbol{z}}) \left(\frac{\partial}{\partial x} A_x \right) [\text{ add \& subtract } \; n_x \hat{\boldsymbol{x}} \frac{\partial}{\partial x} A_x] \\
&= n_x (\boldsymbol{\nabla}) A_x - (\boldsymbol{\nabla}.\boldsymbol{A}) \hat{\boldsymbol{n}} \;\; [\text{see Eq(1)}] \\
\therefore [(\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}) \times \boldsymbol{A}] .\boldsymbol{B} &= \boldsymbol{B}. [n_x (\boldsymbol{\nabla}) A_x - (\boldsymbol{\nabla}.\boldsymbol{A}) \hat{\boldsymbol{n}}] \\
&= n_x (\boldsymbol{B}.\boldsymbol{\nabla}) A_x - (\boldsymbol{\nabla}.\boldsymbol{A}) (\boldsymbol{B}.\hat{\boldsymbol{n}}) \\
&= [\hat{\boldsymbol{n}}.\hat{\boldsymbol{x}}] (\boldsymbol{B}.\boldsymbol{\nabla}) A_x - (\boldsymbol{\nabla}.\boldsymbol{A}) (\boldsymbol{B}.\hat{\boldsymbol{n}}) \\
&= \hat{\boldsymbol{n}}. (\boldsymbol{B}.\boldsymbol{\nabla}) A_x \hat{\boldsymbol{x}} - (\boldsymbol{\nabla}.\boldsymbol{A}) (\boldsymbol{B}.\hat{\boldsymbol{n}}) \\
&= \hat{\boldsymbol{n}}.[(\boldsymbol{B}.\boldsymbol{\nabla}) \boldsymbol{A}] - (\boldsymbol{\nabla}.\boldsymbol{A}) (\boldsymbol{B}.\hat{\boldsymbol{n}}) \\
\text{or } [(\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}) \times \boldsymbol{A}] .\boldsymbol{B} &= \hat{\boldsymbol{n}}. [(\boldsymbol{B}.\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{\nabla}.\boldsymbol{A})\boldsymbol{B}] \;\; [\text{insert this in Eq.(28)}]
\end{aligned}
$$

This completes the proof. For $\boldsymbol{A} = \textit{constant}$ we get Eq(27) again.

Apart from these identities, two further identities, called **Green's Identities** involving two scalar fields $f(\boldsymbol{r})$ & $g(\boldsymbol{r})$ are also important. Green's first identity is easily obtained by applyng the divergence theorem for the vector field $f\boldsymbol{\nabla}g$ & applying the product rule Eq.11:

$$\int_\tau \left(f\,\boldsymbol{\nabla}^2 g + \boldsymbol{\nabla} f.\boldsymbol{\nabla} g\right) d\,\tau = \oint_\sigma f\,\boldsymbol{\nabla} g.\hat{\boldsymbol{n}}\,d\,\sigma \tag{29}$$

The second identity is derived from the 3-d version of the following identity:

$$\frac{d}{d\,x}\left[f(x)\,\frac{d}{d\,x}g(x) - g(x)\,\frac{d}{d\,x}f(x)\right] = f(x)\,\frac{d^2}{d\,x^2}g(x) - g(x)\,\frac{d^2}{d\,x^2}f(x),$$

which is

$$\boldsymbol{\nabla}.\left[f\,\boldsymbol{\nabla} g - g\,\boldsymbol{\nabla} f\right] = f\,\boldsymbol{\nabla}^2 g - g\,\boldsymbol{\nabla}^2 f \tag{30}$$

Applying the divergence theorem yields Green's second identity:

$$\int_\tau \left(f\,\boldsymbol{\nabla}^2 g - g\,\boldsymbol{\nabla}^2 f\right) d\,\tau = \oint_\sigma \left[f\,\boldsymbol{\nabla} g - g\,\boldsymbol{\nabla} f\right].\hat{\boldsymbol{n}}\,d\,\sigma \tag{31}$$