Khatra Adibasi Mahavidyalaya: Lecture Notes

Dr. Siddhartha Sinha

Convergence of Mc-Laurin Series

Power-series

A *real* power-series about a point x_0 is an infinite series in postive powers of $(x - x_0)$, for $x,\, x_0 \in \mathbb{R}$, in the form

$$\sum_{n=0}^{\infty} a_n\,(x - x_0)^n\,, a_n \in \mathbb{R},\ \text{for all } n.$$

Each power series has an *interval of convergence*, centered at the point x_0. If the half-width of the interval is r, with $0 \leq r < \infty$, the power series converges *absolutely* for any point *inside* the interval, *i.e.* for $x_0 - r < x < x_0 + r$ (or $|x - x_0| < r$) & diverges *absolutely* for $|x - x_0| > r$. The convergence behaviour at the boundaries $x_0 \pm r$ is not certain & has to be individually checked. r is called the *radius of convergence* of the given power-series.

A given set of coefficients define a particular power-series, whose sum is a function of x, for $|x - x_0| < r$. Hence, for

$$f(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n\,,\ |x - x_0| < r$$

we say that the series represents $f(x)$ in the interval of convergence & is the power-series **expansion** of $f(x)$ about $x = x_0$.

There are two types of problems involving power-series:

1. given a series, *i.e.*, given a set of coefficients a_n, to find the properties of $f(x)$ &
2. given a function $f(x)$, to find its power series representation, *i.e.*, to find the set of coefficients a_n.

For problem 1, some properties of $f(x)$ inside the interval of convergence are its continuity & infinite times differentiability. This follows readily from the same properties of each term $(x - x_0)^n$. All derivatives of $f(x)$ are themselves convergent power series in the interval of convergence. For example,

$$f^{(1)}(x) = \sum_{n=0}^{\infty} n\,a_n\,(x - x_0)^{n-1} = \sum_{n=1}^{\infty} n\,a_n\,(x - x_0)^{n-1} = \sum_{n'=0}^{\infty} (n' + 1)\,a_{n'+1}\,(x - x_0)^{n'}\,,$$

$$(\text{with}\,\mathrm{n}' = \mathrm{n} - 1);\ = \sum_{n=0}^{\infty} (n + 1)\,a_{n+1}\,(x - x_0)^n\ ,\ \text{with}\,\mathrm{n}' = \mathrm{n}$$

The first term vanishes in the second step, for $n=0$. Also note that the summation index in the first & last step are not equal, even though they have the same symbol.

Similarly, for the k-th derivative,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=0}^{\infty} n(n-1)(n-2)...(n-\overline{k-1})\, a_n\,(x-x_0)^{n-k} \\ &= \sum_{n=k}^{\infty} n(n-1)(n-2)...(n-k+1)\, a_n\,(x-x_0)^{n-k} \\ &= \sum_{n'=0}^{\infty} (n'+k)(n'+k-1)(n'+k-2)...(n'+1)\, a_{n'+k}\,(x-x_0)^{n'} \\ &\quad (\text{with}\, n'=n-k:\ \ n=n'+k,\ n-1=n'+k-1,\ n-2=n'+k-2\,...) \\ &= \sum_{n=0}^{\infty} (n+k)(n+k-1)(n+k-2)...(n+1)\, a_{n+k}\,(x-x_0)^{n} \end{aligned}$$

The derivatives *at* the point of expansion x_0 are of special importance & are obtained by simply substituting $x=x_0$ in the above formula & noting that only the $n=0$ term survives.

$$f^{(k)}(x_0) = k(k-1)(k-2)\,....\,3\,.2\,.1\,a_k = k!\,a_k \tag{1}$$

Taylor Polynomials of a function

We now consider problem 2. Suppose $f(x)$ is infinitely differentiable in an open interval centered at x_0. Then, for any x, we can construct a *polynomial* of order n in powers of $(x-x_0)$ whose k-th coefficient is given by

$$\begin{aligned} a_k &= \frac{f^{(k)}(x_0)}{k!},\ k\le n \\ &= 0,\ k>n \end{aligned} \tag{2}$$

The polynomial $P_n(x)$ of order n constructed in this way is called the *Taylor polynomial approximation* of degree n for $f(x)$ about x_0.

$$P_n(x) = \sum_{k=0}^{n} a_k\ (x-x_0)^k\,,\ \ (a_k\,\text{given by Eq.(2).}) \tag{3}$$

For example, the Taylor polynomials upto degree 3 for e^x about $x=1$ are:

$$\begin{aligned} P_0(x) &= e \\ P_1(x) &= e\,[1+(x-1)] \\ P_2(x) &= e\left[1+(x-1)+(x-1)^2/2\right] \\ P_3(x) &= e\left[1+(x-1)+(x-1)^2/2+(x-1)^3/6\right] \end{aligned}$$

From the above example, & also from Eqn.(3) & (2), $P_n(x_0)=a_0=f(x_0)$. Also, from Eq(1) & Eq(2),

$$\begin{aligned} P_n^{(k)}(x_0) &= k!\,a_k = k!\left(\frac{f^{(k)}(x_0)}{k!}\right) = f^{(k)}(x_0),\ \text{for}\,k\le n \\ &= 0 \ne f^{(k)}(x_0),\ \text{for}\,k>n \end{aligned} \tag{4}$$

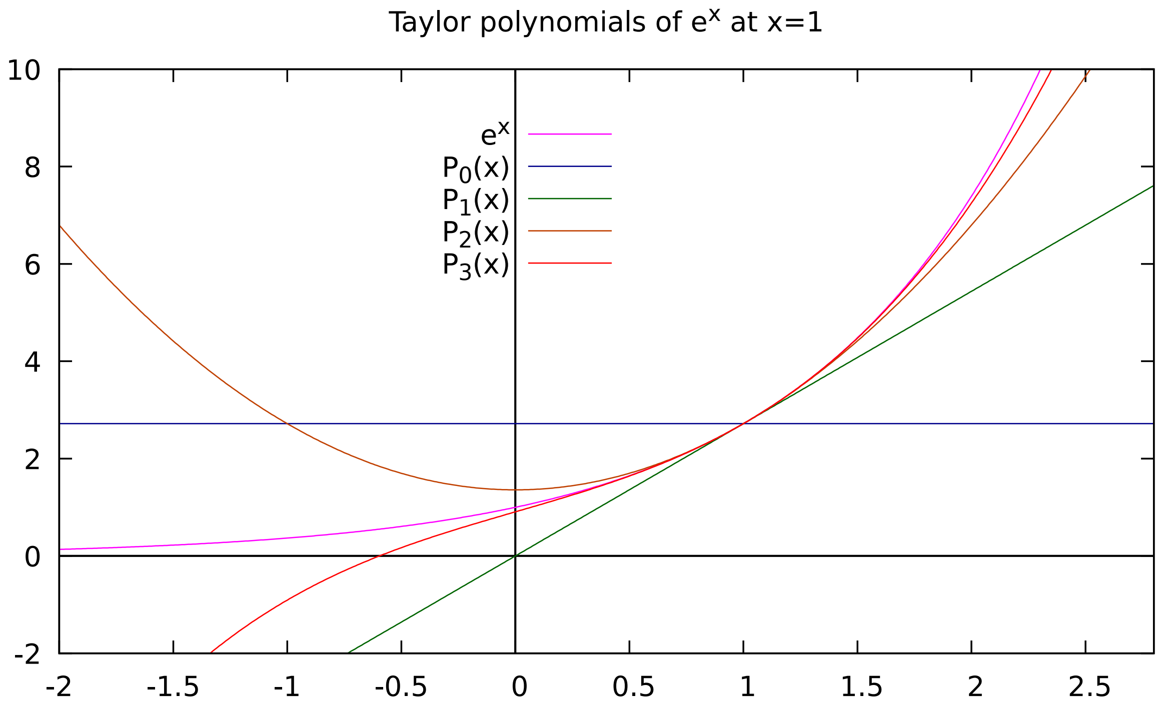

Figure 1: e^x & its first four Talyor polynomial approximatons are plotted in an interval about $x_0 = 1$. The order of equality increases with the order of the polynomial & deviation from e^x decreases as a result near $x_0 = 1$.

In other words, not only $P_n(x_0) = f(x_0)$, but also all derivatives upto order n are equal at the expansion point x_0. Thus, $P_n(x)$ & $f(x)$ are equal at x_0 **upto order n**.

[For any two functions $f(x)$ & $g(x)$, such that $f(x_0) = g(x_0)$, their equality at x_0 is of order n if (or their *inequality* is of order $n + 1$)

$$lim_{x->x_0} = \frac{f(x) - g(x)}{(x - x_0)^k} = 0,\ \ k \leq n$$
$$\neq 0,\ \ k = n + 1.]$$

A *higher* order of equality at x_0 implies a lesser difference in the *neighbourhood* of x_0, as is evident from Fig(1). Hence, for $n > m$, $P_n(x)$ is a better *approximation* to $f(x)$ than $P_m(x)$ in the neighbourhood of x_0.

$$S(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

If $x \in [a, b]$, then $S(x)$ **converges** to $f(x)$ at the given value of x. Hence, we can write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n\ ,\ \ \text{for}\ x \in [a, b] \tag{5}$$

The Mc-Laurin expansion is a special case of Taylor expansion about $x_0 = 0$. Thus, the Mc.Laurin expansion of $f(x)$ is the series

$$S'(x) = f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \ ...$$

Again, it is assumed that $f(x)$ is infinitely differentiable in an interval $[-r, r]$, $r > 0$. r is the *radius of convergence* of $S'(x)$.

For $|x| < r$, the series converges to $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \ \text{for}\ |x| < r \tag{6}$$